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EXACT SOLUTION OF THE HEAT PROPAGATION EQUATIONS IN A RADIATIVE MEDIUM*

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An exact solution of the one-dimensional plane problem of a point explosion in a stationary radiative medium is given. The relations connecting the absorption coefficient and emission source function with the internal energy and frequency, have a special form which admits of a selfsimilar solution.

A problem of a point explosion in a radiative gas taking the motion into account, was studied in /1, 2/.

The equations describing radiative heat transfer in a stationary medium in the one-dimensional plane case have the form

$$\mu\frac{\partial I}{\partial x}+k(I-B)=0,\quad B=B(e,\nu),\quad k=k(e,\nu) \tag{1}$$

$$\frac{\partial e}{\partial t}+\frac{\partial q}{\partial x}=0,\quad q=2\pi\int_0^\infty\int_{-1}^1 I\mu\,d\mu\,d\nu$$

Here e is the internal energy per unit volume, ν is the radiation frequency, I is radiation intensity, k is the absorption coefficient, B is the radiation source function, q is the radiant flux, μ is the cosine of the angle between the direction of the light ray and the x axis, and t is the time.

We shall consider the problem of a point explosion, i.e. we shall seek a solution of (1) satisfying the following initial and boundary conditions:

$$t=0:\ x>0,\ e=0;\quad 2\int_0^\infty e\,dx=E_0 \tag{2}$$

$$t>0:\ x=0,\ \infty;\ q=0$$

The boundary conditions ensure the constancy of the total energy E_0, which is assumed given.

Let the relations connecting k and B with e and ν, have the form

$$k=k_0\varphi(\nu)\,e;\ B=B_0\psi(\nu)\,e;\ k_0,\ B_0=\text{const} \tag{3}$$

In this case the problem formulated above is selfsimilar /2–5/. An analogous problem was studied in /4/ in the diffusion approximation for a grey medium, but the system of selfsimilar equations obtained for this case could not be completely integrated. In case /3/ an exact solution can be obtained for the diffusion approximation, as well as for the exact equation of radiative transfer (1).

We note that relations (3) have a physical meaning. According to Kirchhoff's law B is the radiation function of a perfectly black body

$$B=\frac{2h\nu^3}{c^2\left[\exp\left(h\nu/KT\right)-1\right]}$$

where h is Planck's constant, K is Boltzmann's constant, c is velocity of light and T is temperature. If the temperature is sufficiently high so that $h\nu/(KT)\ll 1$, then $B\approx 2\nu^2KT/c^2$ (the Rayleigh-Jeans law) which agrees with (3).

Let us introduce the selfsimilar variables

$$\lambda=x/(B_0t),\ E(\lambda)=ek_0B_0t,\ J(\lambda,\mu,\nu)=Ik_0t$$

The initial system has the form

$$Q'=E+E'\lambda,\quad Q=2\pi\int_0^\infty\int_{-1}^1 J\mu\,d\mu\,d\nu,\ \mu\frac{\partial J}{\partial\lambda}+\varphi E(J-\psi E)=0 \tag{4}$$

Its solution must satisfy the conditions which follow from (2)

$$E(\infty)=0,\quad Q(0)=Q(\infty)=0,\quad \int_0^\infty E\,d\lambda=\frac{k_0E_0}{2}=\tau_1$$

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Integrating the first equation of (4) taking the condition $Q(0)=0$ at the centre into account, we obtain $Q=E\lambda$. Let us introduce the new variable

$$\tau=\int_0^\lambda E\,d\lambda,\quad 0\leqslant\tau\leqslant\tau_1$$

Finally, from (4) we obtain

$$E=\frac{dQ}{d\tau}E-Q\frac{dE}{d\tau},\quad \mu\frac{\partial J}{\partial\tau}+\varphi(J-\psi E)=0 \tag{5}$$

$$E(\tau_1)=0,\quad Q(0)=Q(\tau_1)=0,\quad Q=2\pi\int_0^\infty\int_{-1}^1 J\mu\,d\mu\,d\nu$$

If we do not resort to angular averaging of the transport equation, the integral boundary conditions $Q(0)=Q(\tau_1)=0$ ensuring conservation of the energy of the explosion will, generally speaking, be insufficient to obtain a unique solution. Let the condition of specular reflection $J(\mu)=J(-\mu)$ be satisfied at the boundaries $\tau=0$ and $\tau=\tau_1$. Then the solution sought will have the form

$$E=A(1+\cos\alpha),\quad Q=4\pi AC\sin\alpha,\quad \lambda=\frac{Q}{E} \tag{6}$$

$$J=A\psi\left(1+\frac{\tau_2^2\cos\alpha+\tau_2\mu\sin\alpha}{\tau_2^2+\mu^2}\right),\quad \tau_2=\frac{\tau_1\varphi}{\pi} \tag{7}$$

$$\alpha=\frac{\tau\pi}{\tau_1},\quad A=\frac{\tau_1}{4\pi^2C},\quad C=\int_0^\infty\psi\tau_2(1-\tau_2\operatorname{Arcctg}\tau_2)\,d\nu$$

Eliminating α, we finally obtain

$$E=2A\left[1+\left(\frac{\lambda}{4\pi C}\right)^2\right]^{-1} \tag{8}$$

In the diffusion approximation, (5) will be replaced by

$$E=Q'E-QE',\quad Q=\int_0^\infty V\,d\nu,\quad U=2\pi\int_{-1}^1 J\,d\mu$$
$$V'+\varphi(U-4\pi\psi E)=0,\quad U'+3\varphi V=0$$
$$E(\tau_1)=Q(0)=Q(\tau_1)=0$$

The solution is obtained in the same functional form (6), where A, C are replaced by A_1, C_1 and

$$U=4\pi A_1\psi\left(1+\frac{3\tau_2^2\cos\alpha}{3\tau_2^2+1}\right),\quad V=4\pi A_1\psi\,\frac{\tau_2\sin\alpha}{3\tau_2^2+1}$$

$$A_1=\frac{\tau_1}{4\pi^2C_1},\quad C_1=\int_0^\infty\psi\,\frac{\tau_2}{3\tau_2^2+1}\,d\nu$$

Eliminating α, once again, replacing A, C by A_1, C_1, we arrive expression (8). It can be shown that the diffusion approximation yields a solution which is almost exact, since $C\approx C_1$.

We note that the nature of the selfsimilarity and the form of solution (8) are analogous to the solution of the corresponding problem for the equation of heat conduction

$$\frac{\partial e}{\partial t}=\frac{\partial}{\partial x}\left(\varkappa\frac{\partial e}{\partial x}\right),\quad \varkappa=\frac{\varkappa_0}{e},\quad \varkappa_0=\text{const}$$

Analysis shows that by putting $\varkappa_0=2B_0E_0C$ we can obtain complete agreement between the solutions.

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